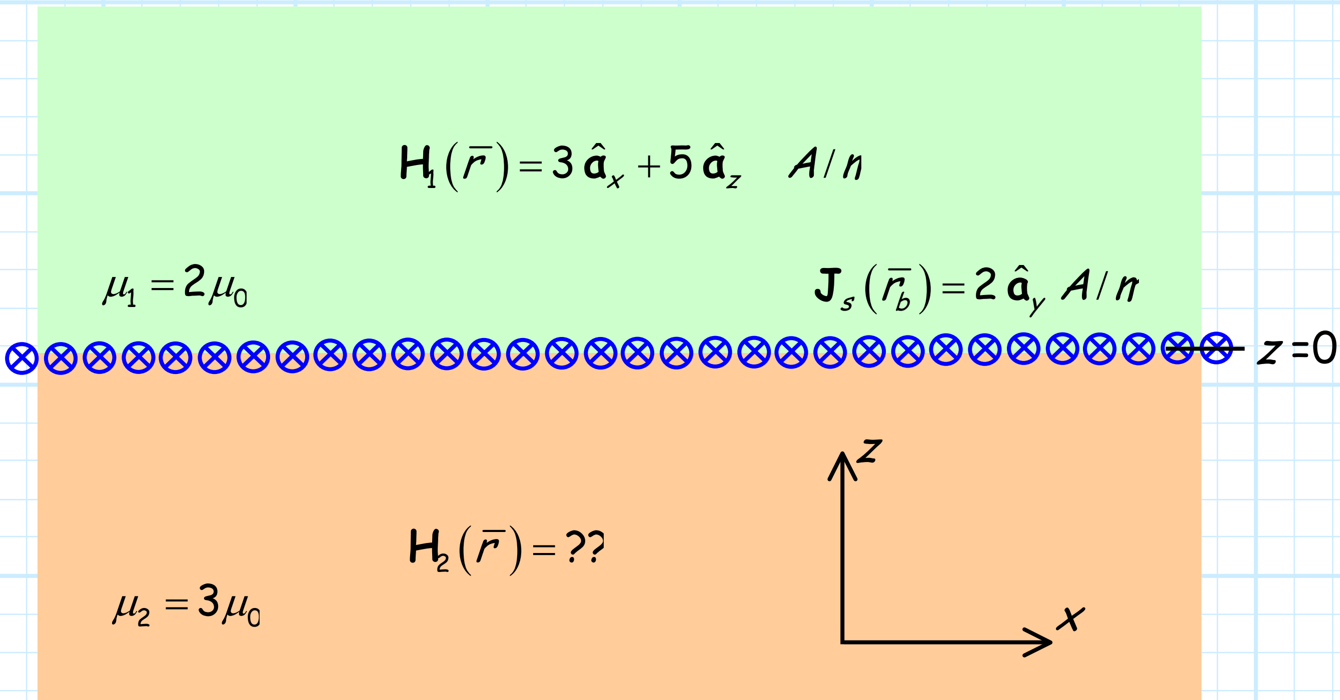
Example: Magnetostatic Boundary Conditions

Consider two **magnetic** materials, separated by some **boundary**:



Throughout region 1, there is a **constant** magnetic field:

$$\mathbf{H}_1(\bar{r}) = 3\,\hat{\mathbf{a}}_x + 5\,\hat{\mathbf{a}}_z \quad (z > 0)$$

On the **interface** (i.e., boundary) between the two regions, there flows a **surface current**:

$$\mathbf{J}_s(\bar{r}) = \begin{cases} 0 & z > 0 \\ 2\,\hat{\mathbf{a}}_y & z = 0 \\ 0 & z < 0 \end{cases} \qquad \left[A/m\right]$$

Q: *What is* $\mathbf{H}_2(\bar{r})$ *and* $\mathbf{B}_2(\bar{r})$ *in region 2 ??*

A: Let's apply the **boundary conditions** and find out!

At the interface (i.e., z=0), we can state that:

$$\mathbf{H}_2(\bar{r}_b) = \mathbf{H}_2(z=0) = H_{2x}(z=0)\,\hat{\mathbf{a}}_x + H_{2y}(z=0)\,\hat{\mathbf{a}}_y + H_{2z}(z=0)\,\hat{\mathbf{a}}_z$$

and:

$$\mathbf{B}_2(\bar{r}_b) = \mathbf{B}_2(z=0) = B_{2x}(z=0)\,\hat{\mathbf{a}}_x + B_{2y}(z=0)\,\hat{\mathbf{a}}_y + B_{2z}(z=0)\,\hat{\mathbf{a}}_z$$

Therefore, we need to find the **scalar components** $H_{2x}(z=0)$, $B_{2x}(z=0)$, etc.

First, we note that $\hat{\mathbf{a}}_z$ is **normal** to the interface, while $\hat{\mathbf{a}}_y$ and $\hat{\mathbf{a}}_z$ are **tangential**.

Thus, from **boundary condition**:

$$\hat{a}_n \times \left(\mathbf{H}_1(\bar{r}_b) - \mathbf{H}_2(\bar{r}_b) \right) = \mathbf{J}_s(\bar{r}_b)$$

where we note that $\hat{\mathbf{a}}_n = \hat{\mathbf{a}}_z$, we find:

$$\begin{aligned}
\hat{\mathbf{a}}_z \times \left(\mathbf{H}_1(z=0) - \mathbf{H}_2(z=0) \right) &= \mathbf{J}_s(z=0) \\
\hat{\mathbf{a}}_z \times \left[(3 - H_{2x})\hat{\mathbf{a}}_x + (0 - H_{2y})\hat{\mathbf{a}}_y + (5 - H_{2z})\hat{\mathbf{a}}_z \right] &= 2\,\hat{\mathbf{a}}_y \\
(3 - H_{2x})\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_x + (0 - H_{2y})\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_y + (5 - H_{2z})\hat{\mathbf{a}}_z \times \hat{\mathbf{a}}_z &= 2\,\hat{\mathbf{a}}_y \\
(3 - H_{2x})\hat{\mathbf{a}}_y - (0 - H_{2y})\hat{\mathbf{a}}_x &= 2\,\hat{\mathbf{a}}_y \\
(3 - H_{2x})\hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x &= 2\,\hat{\mathbf{a}}_y
\end{aligned}$$

Thus, we can ascertain:

$$\begin{aligned}
(3 - H_{2x})\hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x &= 2\,\hat{\mathbf{a}}_y \\
(3 - H_{2x})\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_x + H_{2y}\hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_x &= 2\,\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_x \\
H_{2y} &= 0
\end{aligned}$$

and likewise:

$$\begin{aligned}
(3 - H_{2x})\hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x &= 2\,\hat{\mathbf{a}}_y \\
(3 - H_{2x})\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_y + H_{2y}\hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_y &= 2\,\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_y \\
3 - H_{2x} &= 2 \\
H_{2x} &= 1
\end{aligned}$$

Therefore:

$$H_{2x}(z=0) = 1 \qquad \text{and} \qquad H_{2y}(z=0) = 0$$

Q: *But what about scalar component* $H_{2z}(z=0)$*?*

A: We can find it using our **second** boundary condition:

$$\mu_1 \mathbf{H}_{1n}(\bar{r}_b) = \mu_2 \mathbf{H}_{2n}(\bar{r}_b)$$

From which we find:

$$\mu_1 H_{1z}(z=0)\hat{\mathbf{a}}_z = \mu_2 H_{2z}(z=0)\hat{\mathbf{a}}_z$$

$$2\mu_0\, 5\,\hat{\mathbf{a}}_z = 3\mu_0\, H_{2z}(z=0)\,\hat{\mathbf{a}}_z$$

And therefore:

$$2\mu_0\, 5\,\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z = 3\mu_0\, H_{2z}(z=0)\,\hat{\mathbf{a}}_z \cdot \hat{\mathbf{a}}_z$$

$$2\mu_0\, 5 = 3\mu_0\, H_{2z}(z=0)$$

$$H_{2z}(z=0) = \frac{10}{3}$$

Thus, we find that:

$$\begin{aligned}\mathbf{H}_2(z=0) &= H_{2x}(z=0)\,\hat{\mathbf{a}}_x + H_{2y}(z=0)\,\hat{\mathbf{a}}_y + H_{2z}(z=0)\,\hat{\mathbf{a}}_z \\ &= \hat{\mathbf{a}}_x + {}^{10}\!/_{3}\,\hat{\mathbf{a}}_z\end{aligned}$$

And since:

$$\mathbf{B}_2(z=0) = \mu_2\,\mathbf{H}_2(z=0)$$

We find:

$$\begin{aligned}\mathbf{B}_2(z=0) &= 3\mu_0\left(\hat{\mathbf{a}}_x + 10/3\,\hat{\mathbf{a}}_z\right)\\ &= 3\mu_0\,\hat{\mathbf{a}}_x + 10\mu_0\,\hat{\mathbf{a}}_z\end{aligned}$$

Q: *But these are the values of the fields at the interface—what are the fields* ***throughout*** *region 2?*

A: Note that there are **no conduction currents** within region 2. Thus, we find **within region 2**:

$$\nabla\times\mathbf{H}_2(\bar{r}) = 0 \qquad (z<0)$$

Note that a **constant** magnetic field will satisfy the above equation. Moreover, the following **constant** magnetic field will **likewise** satisfy our **boundary condition** $\mathbf{H}_2(z=0)$:

$$\mathbf{H}_2(\bar{r}) = \hat{\mathbf{a}}_x + \tfrac{10}{3}\,\hat{\mathbf{a}}_z \qquad \left[\tfrac{A}{m}\right]$$

In other words, the value of the magnetic field at the boundary is **likewise** the value of the magnetic field **everywhere throughout** region 2 ($\mathbf{H}_2(\bar{r})$ is a **constant** vector field!). The **magnetic flux density** is therefore:

$$\mathbf{B}_2(\bar{r}) = \mu_2\,\mathbf{H}_2(\bar{r}) = 3\mu_0\,\hat{\mathbf{a}}_x + 10\mu_0\,\hat{\mathbf{a}}_z \qquad \left[W/m^2\right]$$